

ASYMPTOTIC ORDERS OF REACHABILITY**IN PERTURBED LINEAR SYSTEMS¹**Cüneyt M. Özveren²Alan S. Willsky²George C. Verghese³**Abstract**

A framework for studying asymptotic orders of reachability in perturbed linear, time-invariant systems is developed. The systems of interest are defined by matrices that have Taylor or Laurent expansions in the perturbation parameter ϵ about the point 0. The reachability structure is exposed via the Smith form of the reachability matrix. This approach is used to provide insight into the kinds of inputs needed to reach weakly reachable target states, into the structure of high-gain feedback for pole placement, and into the types of inputs that steer trajectories arbitrarily close to almost (A, B) -invariant subspaces and almost (A, B) -controllability subspaces.

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²Department of Electrical Engineering and Computer Science and Laboratory for Information and Decision Systems, MIT, Cambridge, MA 02139.

³Department of Electrical Engineering and Computer Science and Laboratory for Electromagnetic and Electronic Systems, MIT, Cambridge, MA 02139.

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I. INTRODUCTION

I.1 MOTIVATION

In this paper, we develop and apply a theory of asymptotic orders of reachability in linear time-invariant systems parametrized by some small variable, ϵ . To provide a motivation for the key issues in our approach, consider the following discrete time system as an example:

Example 1.1

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ .01 & 2 \end{bmatrix}x[k] + \begin{bmatrix} 1 \\ .01 \end{bmatrix}u[k]$$

This system is reachable but the reachability matrix

$$[b | Ab] = \begin{bmatrix} 1 & 1.01 \\ .01 & .03 \end{bmatrix}$$

is not very far from a singular matrix, in that its condition number is approximately 10^4 . This leads to numerical difficulties in determining reachability, as shown in [3]. Also, consider the minimum energy control problem for this system. The minimum energy control to reach $x[2] = [1 \ 0]'$ (where ' denotes the transpose) from $x[0] = 0$ is $u_1[1] = -.5$ and $u_1[2] = 1.5$, while the minimum energy control for $x[2] = [1 \ 1]'$ is $u_2[1] = 49.7$ and $u_2[2] = -49$. This order of magnitude difference between u_1 and u_2 is another indication of near unreachability. Still further indications may be obtained, for example by considering how small a perturbation of the system matrices suffices to destroy reachability (in this case, 0.01), or by examining the magnitude of feedback gain required to shift poles by various amounts (in

this case, to move the eigenvalues by 2, feedback gains of magnitude approximately 10^2 are required, as illustrated in Example 3.1). ..

Our treatment of problems of this type is qualitative rather than numerical in nature: we assume that small values in the system are modeled by functions of a small parameter ϵ , which implicitly indicates the presence of different orders of coupling among state variables and inputs. Parametrized linear systems are studied in general by Kamen and Khargonekar [13] and Brewer et al. [14]. However, we look at how unreachable the system is in terms of "orders of ϵ ". Specifically, we consider continuous time and discrete time systems of the form

$$\dot{x}(t) = A(\epsilon)x(t) + B(\epsilon)u(t) \quad (1.1)$$

$$x[k+1] = A(\epsilon)x[k] + B(\epsilon)u[k] \quad (1.2)$$

where $A(\epsilon)$ and $B(\epsilon)$ have Laurent expansions around $\epsilon=0$:

$$A(\epsilon) : \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon)) \quad (1.3)$$

$$B(\epsilon) : \mathbb{R}^m((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon)) \quad (1.4)$$

(We write $a(\epsilon) \in \mathbb{R}((\epsilon))$ if $a(\epsilon)$ has a Laurent expansion around $\epsilon=0$.) Defining these systems over $\mathbb{R}((\epsilon))$ permits us to examine the effect or necessity of high gain feedback.

This work was particularly motivated by the numerical problems encountered in various pole placement methods and in evaluating system reachability. Pole placement and related numerical issues are addressed using various approaches in the

current literature [4-7]. In multi-input systems, unlike single-input systems, the feedback matrix that produces a given set of poles is not unique, and the additional degrees of freedom may be used to attain other control objectives (see [7]). One may, for example, attempt to minimize the maximum feedback gain; [5] addresses this problem via numerical examples on redistribution of the feedback task among the inputs and balancing the A and B matrices. These examples contain some intuitive ideas, but have not led to systematic procedures that work well for well-defined and substantial classes of systems. One of our objectives here is to suggest an analytical approach to understanding and structuring feedback gains for pole placement.

Another area of numerical work involves criteria to measure controllability. Boley and Lu [9] use the "distance to the nearest uncontrollable system" as a criterion. They define this by the minimum norm perturbation that would make a system uncontrollable. They also relate this concept to state feedback by measuring the amount that the eigenvalues move due to state feedback of bounded magnitude. Connections may also be made to the literature on balanced realizations, [8], where the singular values of the controllability Grammian are used to indicate nearness to uncontrollability.

The issue of controllability in perturbed systems of the form (1.1) has been examined by Chow [15]. He defines a system to be strongly controllable if the system is controllable at $\epsilon = 0$. Otherwise, he calls it weakly controllable and concludes that pole

placement of such systems will require controls with large gains. Chow looks at systems with two time scales (slow and fast), and he proves that a necessary and sufficient condition for such a 'singularly perturbed' system to be strongly controllable is the controllability of its slow and fast subsystems.

Our analysis goes further than Chow's in that we examine the relative orders of reachability of different parts of the state space. The methods we use have some similarity to those used by Lou et al. [1,2], who relate the multiple time scale structure of the system (1.1) to the invariant factors of $A(\epsilon)$, when this matrix has entries from the ring of functions analytic at $\epsilon = 0$. The Smith decomposition of $A(\epsilon)$ plays a key role in their analysis, while the Smith decomposition of the reachability matrix is central to the development in this paper. While the primary focus of the work in [1,2] is on time scale structure, some attention is paid to control. In particular, [1] gives results on the use of feedback in (1.1) to change the time scale structure of the system. The work in [22] may be seen as a continuation of the work in [1,2] in that it analyzes the effect of control and feedback on the system of (1.1). This paper is based on the work in [22].

I.2 OUTLINE

In Section II, we develop a theory of orders of reachability. We start with discrete time systems and illustrate that the orders of reachability can be recovered from the Smith decomposition of

the reachability matrix. We define a standard form which displays these orders explicitly. Also, we show that equivalent results hold for continuous time systems. In Section III, this theory is extended to pole placement by full state feedback for systems whose entries have Taylor expansions around $\epsilon=0$. We also provide a computationally efficient and numerically well-behaved algorithm for pole placement. Section IV develops connections with Willems' work on "almost invariance" [3]. We show that the subspace a sequence of (A, B) -controllability subspaces converge to is almost (A, B) -invariant. In Section V, we summarize our results and suggest problems for further research.

I.3 ASSUMPTIONS

The reachability matrix for the systems in (1.1) and (1.2) is $\mathcal{C}(\epsilon) = [B(\epsilon) | A(\epsilon)B(\epsilon) | \dots | A^{n-1}(\epsilon)B(\epsilon)] : \mathbb{R}^{mn}((\epsilon)) \rightarrow \mathbb{R}^n((\epsilon))$. We assume that the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$, i.e. they have Taylor expansions around $\epsilon=0$. We shall show (Proposition 2.6) that this is equivalent to the system being what we term a proper system. This is not a restrictive condition for continuous time systems since it can be achieved by time scaling. However, it is a restrictive assumption for discrete time systems.

Note that $\mathcal{C}(\epsilon)$ can be made analytic at $\epsilon = 0$ (i.e. made into a matrix over $\mathbb{R}[[\epsilon]]$) by a simple input scaling, and this will be done when convenient. In addition, we assume that the reachability matrix is full row rank for all $\epsilon \in (0, a)$, $a \in \mathbb{R}^+$. In

the cases of most interest to us, the reachability matrix will lose rank for $\epsilon = 0$, and α will be the smallest positive value of ϵ for which the reachability matrix loses rank. Under these conditions, we analyze the asymptotic reachability of the system as $\epsilon \downarrow 0$.

II. ORDERS OF REACHABILITY

II.1 ϵ^j -REACHABILITY

We start by developing our theory of asymptotic orders of reachability in an analogous way to existing linear control theory. In order to provide a motivation for our approach, let us start with the following counterpart of Example 1.1:

Example 2.1:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ \epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

so

$$\Phi(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & 3\epsilon \end{bmatrix}$$

This system is reachable for all $\epsilon \in (0, 2)$. The minimum energy control sequence needed to go from the origin to $x_1[2] = [1 \ 0]'$ is $u_1[1] = -1/(2-\epsilon)$ and $u_1[2] = 3/(2-\epsilon)$, which is $O(1)$.⁽⁴⁾ The minimum energy control sequence for $x_2[2] = [1 \ 1]'$ is $u_2[1] = (-\epsilon+1)/\epsilon(2-\epsilon)$ and $u_2[2] = (2\epsilon-1)/\epsilon(2-\epsilon)$, which is $O(1/\epsilon)$. ..

We next generalize this characterization of target states by the order of control sufficient to reach them.

Definition 2.2: $x(\epsilon) \in \mathbb{R}^n[[\epsilon]]$ is ϵ^j -reachable if there exists an $O(1/\epsilon^j)$ input sequence $u(\epsilon) \equiv [u'[n-1] \cdots u'[0]]'$ such that $x(\epsilon)$

⁴ $f(\epsilon)$ is $O(\epsilon^k)$ if $\lim_{\epsilon \downarrow 0} \|f(\epsilon)\|/\epsilon^k$ exists, where k is an integer, $f(\epsilon)$ is a scalar, vector or matrix, and $\|*\|$ denotes the appropriate norm. Note that if $f(\epsilon)$ is $O(\epsilon^k)$ then it is also $O(\epsilon^{k-1})$, $O(\epsilon^{k-2})$ etc.

is reached from zero in n steps using $\mathcal{U}(\epsilon)$ (i.e. $x(\epsilon) = \mathcal{C}(\epsilon)\mathcal{U}(\epsilon)$). Let \mathcal{X}^j be the set of all ϵ^j -reachable states, then $\mathcal{X}^0 \subset \mathcal{X}^1 \subset \mathcal{X}^2 \subset \dots$ and \mathcal{X}^j is an $\mathbb{R}[[\epsilon]]$ -submodule of $\mathbb{R}^n[[\epsilon]]$. We term \mathcal{X}^j the ϵ^j -reachable submodule.

..

Note that if $x(\epsilon)$ is ϵ^j -reachable, then $(1/\epsilon)x(\epsilon)$ is not necessarily ϵ^j -reachable. Thus if we had considered target states in $\mathbb{R}^n((\epsilon))$ in Definition 2.2, then the set of ϵ^j reachable states would not be $\mathbb{R}((\epsilon))$ -subspaces.

In Example 2.1, $\mathcal{X}^0 = \text{Im}[1 \ 0]^\top + \epsilon\mathbb{R}^2[[\epsilon]]$, $\mathcal{X}^1 = \mathcal{X}^2 = \dots = \mathbb{R}^2[[\epsilon]]$.

An interesting property of the set of ϵ^j -reachable submodules is that all the structure is embedded in the ϵ^0 -reachable submodule. First of all, note that \mathcal{X}^0 is the image of the reachability matrix under the set of all control sequence vectors $\mathcal{U}(\epsilon)$ in $\mathbb{R}^{mn}[[\epsilon]]$. Also, the ϵ^j -reachable submodule is simply obtained by scaling the ϵ^{j-1} -reachable submodule by $1/\epsilon$. To state this formally:

Proposition 2.3: $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)\mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$ and $\mathcal{X}^j = \frac{1}{\epsilon} \{ \mathcal{X}^{j-1} \cap \epsilon\mathbb{R}^n[[\epsilon]] \} = \frac{1}{\epsilon^j} \{ \mathcal{X}^{j-i} \cap \epsilon^i\mathbb{R}^n[[\epsilon]] \}$, for nonnegative integers i, j and $j \geq i$.

Proof: By Definition 2.2, $\mathcal{X}^0 = \{\mathcal{C}(\epsilon)\mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$, or in general $\mathcal{X}^j = \{\mathcal{C}(\epsilon)1/\epsilon^j\mathbb{R}^{mn}[[\epsilon]]\} \cap \mathbb{R}^n[[\epsilon]]$. Then,

$$\frac{1}{\epsilon^i} \{ \mathcal{X}^{j-i} \cap \epsilon^i\mathbb{R}^n[[\epsilon]] \} = \frac{1}{\epsilon^i} \{ \{ \frac{1}{\epsilon^{j-i}} \mathcal{C}(\epsilon)\mathbb{R}^{mn}[[\epsilon]] \} \cap \epsilon^i\mathbb{R}^n[[\epsilon]] \}$$

$$= \left\{ \frac{1}{\epsilon^j} \mathcal{C}(\epsilon) \mathbb{R}^{mn}[[\epsilon]] \right\} \cap \mathbb{R}^n[[\epsilon]] = \mathcal{X}^j \quad \dots$$

The structure of the ϵ^j -reachable submodules is not always as easily obtained by inspection of the pair $(A(\epsilon), B(\epsilon))$ as it was in Example 2. To illustrate this, consider an ϵ perturbation of Example 2.1:

Example 2.4:

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ -\epsilon & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} u[k]$$

where

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 1+\epsilon \\ \epsilon & \epsilon \end{bmatrix}$$

This system is reachable for all $\epsilon \in (0, \infty)$. In this case, we find that $x_1[2] = [1 \ 0]'$ is ϵ -reachable, and $x_2[2] = [1 \ 1]'$ is ϵ^2 -reachable. Therefore, even an ϵ perturbation may cause drastic changes in our submodules. \dots

II.2 SMITH DECOMPOSITION OF $\mathcal{C}(\epsilon)$

The key element in our results is the Smith decomposition of $\mathcal{C}(\epsilon)$ since we are interested in how $\mathcal{C}(\epsilon)$ becomes singular as $\epsilon \downarrow 0$. For simplicity, as noted in the Introduction, let us assume that $\mathcal{C}(\epsilon)$ has a Taylor expansion around $\epsilon=0$ and that it is full row rank for all $\epsilon \in (0, a)$, $a \in \mathbb{R}^+$, then the $n \times mn$ matrix $\mathcal{C}(\epsilon)$ has a Smith decomposition [1, 2, 11, 12]

$$\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon) \quad (2.1)$$

where $P(\epsilon)$, $n \times n$, is unimodular ($\det P(0) \neq 0$), $Q(\epsilon)$, $n \times mn$, is full row rank at $\epsilon=0$, and

$$D(\epsilon) = \text{diag}\{I_{p_0}, \epsilon I_{p_1}, \dots, \epsilon^k I_{p_k}\} \quad (2.2)$$

is $n \times n$ where I_{p_i} denotes a $p_i \times p_i$ identity matrix with $p_i = 0$ corresponding to absence of the i -th block, and with $p_k \neq 0$. The indices p_i , and hence $D(\epsilon)$, are unique, though $P(\epsilon)$ and $Q(\epsilon)$ are not.

Now, $x^j = P(\epsilon)y^j$ where

$$y^j = \epsilon_j + \epsilon\epsilon_{j+1} + \dots + \epsilon^{k-1-j}\epsilon_{k-1} + \epsilon^{k-j}R^n[[\epsilon]] \quad (2.3)$$

and $\epsilon_i = \text{Im}[I_{n_i} \ 0]'$, $n_i = p_0 + \dots + p_i$. In fact y^j is the ϵ^j -reachable submodule of the original system similarity transformed by $P(\epsilon)$ and its structure immediately follows from the indices. This property is captured in a standard form defined in the next section.

II.3 STANDARD FORM

Consider a pair $(A(\epsilon), B(\epsilon))$ with a Smith decomposition of its reachability matrix defined as above. We will term such a system an ϵ^k -reachable system with indices n_0, \dots, n_k . Let $\bar{A}(\epsilon) = P^{-1}(\epsilon)A(\epsilon)P(\epsilon)$ and $\bar{B}(\epsilon) = P^{-1}(\epsilon)B(\epsilon)$. The pair $(\bar{A}(\epsilon), \bar{B}(\epsilon))$ will be called a standard form for $(A(\epsilon), B(\epsilon))$.

The system in Example 2.1 is already in standard form. For the system in Example 2.4, a Smith decomposition of the reachability matrix is:

$$\mathcal{C}(\epsilon) = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \begin{bmatrix} 1 & 1+\epsilon \\ 0 & -1 \end{bmatrix} = P(\epsilon)D(\epsilon)Q(\epsilon)$$

Transforming the system by $P(\epsilon)$ yields

$$y[k+1] = \begin{bmatrix} 1+\epsilon & 1 \\ -\epsilon^2 & 2-\epsilon \end{bmatrix} y[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which uncovers the previously hidden ϵ^2 structure.

A standard form of a system is termed a proper standard form if it has the following structure:

$$A(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & 1/\epsilon A_{0,1}(\epsilon) & \dots & 1/\epsilon^k A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & 1/\epsilon^{k-1} A_{1,k}(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1} A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_k\} \quad (2.4a)$$

$$B(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_k\} \quad (2.4b)$$

where $A_{i,j}(\epsilon)$ are analytic at $\epsilon=0$, and $n_i = \sum_{j=0}^i p_j$.

Example 2.1 and the transformed version of Example 2.4 are both in proper standard form. In fact, the next result shows that finding one proper standard form is enough to conclude that all standard forms of a pair are proper:

Proposition 2.5: If a pair $(A(\epsilon), B(\epsilon))$ has a proper standard form, then all standard forms of $(A(\epsilon), B(\epsilon))$ are proper.

Proof: Let $\mathcal{C}(\epsilon) = P_1(\epsilon)D(\epsilon)Q_1(\epsilon) = P_2(\epsilon)D(\epsilon)Q_2(\epsilon)$, then $A_i(\epsilon) = P_i^{-1}(\epsilon)A(\epsilon)P_i(\epsilon)$, $B_i(\epsilon) = P_i^{-1}(\epsilon)B(\epsilon)$ for $i=1,2$ are two standard forms. Suppose that the pair $(A_1(\epsilon), B_1(\epsilon))$ is a proper standard form. Let $\bar{A}_i(\epsilon) = D^{-1}(\epsilon)A_i(\epsilon)D(\epsilon)$, $\bar{B}_i(\epsilon) = D^{-1}(\epsilon)B_i(\epsilon)$ for $i=1,2$. Note $\bar{A}_1(\epsilon)$ and $\bar{B}_1(\epsilon)$ are both over $\mathbb{R}[[\epsilon]]$. We wish to show that the same is true for $\bar{A}_2(\epsilon)$ and $\bar{B}_2(\epsilon)$. Let $R(\epsilon) = D^{-1}(\epsilon)P_2^{-1}(\epsilon)P_1(\epsilon)D(\epsilon)$, then $R(\epsilon)$ is invertible, and $Q_2(\epsilon) = R(\epsilon)Q_1(\epsilon)$. But then $R(\epsilon) = Q_2(\epsilon)Q_1^+(\epsilon)$ and $R^{-1}(\epsilon) = Q_1(\epsilon)Q_2^+(\epsilon)$, where $Q_i^+(\epsilon)$ denotes the right inverse of

$Q_i(\epsilon)$, which exists over $\mathbb{R}[[\epsilon]]$. Thus, $R(\epsilon)$ is unimodular. Since $(\bar{A}_1(\epsilon), \bar{B}_1(\epsilon))$ is over $\mathbb{R}[[\epsilon]]$ and $\bar{A}_2(\epsilon) = R(\epsilon)\bar{A}_1(\epsilon)R^{-1}(\epsilon)$, $\bar{B}_2(\epsilon) = R(\epsilon)\bar{B}_1(\epsilon)$, the pair $(\bar{A}_2(\epsilon), \bar{B}_2(\epsilon))$ is also over $\mathbb{R}[[\epsilon]]$. Therefore, $(A_2(\epsilon), B_2(\epsilon))$ is a proper standard form. ..

A pair $(A(\epsilon), B(\epsilon))$ is termed proper if it has a proper standard form. Thus, both of the systems in Examples 2.1 and 2.4 are proper. Our assumption that the coefficients of the characteristic polynomial of $A(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$ is necessary and sufficient for a system to be proper. In general, we have the following:

Proposition 2.6: The following statements are equivalent for any pair $(A(\epsilon), B(\epsilon))$ such that $\mathcal{C}_n(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$:

1. $(A(\epsilon), B(\epsilon))$ is proper.
2. $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.
3. The coefficients of the characteristic polynomial, $\sigma(A(\epsilon))$, of $A(\epsilon)$ are over $\mathbb{R}[[\epsilon]]$.

To prove this result, let us first consider the following two lemmas:

Lemma 2.7: For a pair $(A(\epsilon), B(\epsilon))$ with $\mathcal{C}_n(\epsilon)$ over $\mathbb{R}[[\epsilon]]$, $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff the coefficients of $\sigma(A(\epsilon))$ are in $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Follows using the Cayley-Hamilton theorem.

(\leftarrow) Suppose not all coefficients of $\sigma(A(\epsilon))$ are in $\mathbb{R}[[\epsilon]]$, then some eigenvalue of $A(\epsilon)$, say $\bar{\lambda}(\epsilon)$, is not $O(1)$. Let the Jordan decomposition of $A(\epsilon)$ in some interval $\epsilon \in (0, a)$ be

$$A(\epsilon) = X^{-1} \Lambda(\epsilon) X(\epsilon), \text{ where } X(\epsilon), \Lambda(\epsilon) \text{ are continuous and } X(\epsilon) \text{ is}$$

scaled such that $\lim_{\epsilon \downarrow 0} X(\epsilon)$ exists. See [23] for the existence of such a decomposition. Consider:

$$X(\epsilon)A^j(\epsilon)\mathcal{C}(\epsilon) = \Lambda^j(\epsilon)X(\epsilon)\mathcal{C}(\epsilon) \quad (2.5)$$

Note that some row of $\Lambda^j(\epsilon)$, say the i^{th} row, has the form $[0 \dots 0 \ \bar{\lambda}^j(\epsilon) \ 0 \dots 0]$, while the i^{th} row of $X(\epsilon)\mathcal{C}(\epsilon)$ is nonzero and hence of finite order in ϵ . Thus, by choosing j large enough, we can obtain a right hand side in (2.5) that is not $O(1)$. It follows that $A^j(\epsilon)\mathcal{C}(\epsilon)$ is not $O(1)$ for large enough j . But $\mathcal{C}_\infty(\epsilon)$ contains the entries of $A^j(\epsilon)\mathcal{C}(\epsilon)$, so $\mathcal{C}_\infty(\epsilon)$ is not $O(1)$. \dots

Lemma 2.8: Let $\bar{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$,

$\bar{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)B(\epsilon)$, then $\mathcal{C}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff $\bar{\mathcal{C}}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Follows from the transformation.

(\leftarrow) Clearly $\bar{\mathcal{C}}_n(\epsilon) = Q(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$, and the rest follows using Lemma 2.3 and the Cayley-Hamilton theorem. \dots

We can now prove Proposition 2.6:

Proof (of Proposition 2.6): $(1 \rightarrow 2)$ Follows from the definition of a proper form and the structure in (2.4).

$(2 \rightarrow 1)$ By Lemma 2.8, $\bar{\mathcal{C}}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$. Consider

$\bar{\mathcal{C}}_{n+1}(\epsilon) = [\bar{B}(\epsilon) \mid \bar{A}(\epsilon)\bar{\mathcal{C}}_n(\epsilon)]$, which is also over $\mathbb{R}[[\epsilon]]$. Then, $\bar{B}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$. Also, $\bar{A}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ since $\bar{\mathcal{C}}_n(\epsilon)$ is full row rank at $\epsilon=0$ and therefore has a right inverse over $\mathbb{R}[[\epsilon]]$. Thus, $(D(\epsilon)\bar{A}(\epsilon)D^{-1}(\epsilon), D(\epsilon)\bar{B}(\epsilon))$ is a proper standard form.

$(2 \leftrightarrow 3)$ Lemma 2.7 \dots

As an immediate consequence of statement 2 of Proposition 2.6 we have the following important property of proper systems:

Corollary 2.9: Given a proper pair $(A(\epsilon), B(\epsilon))$, $x \in \mathcal{X}^j$ iff x is reachable with $O(1/\epsilon^j)$ control in p steps, for all $p \geq n$. ..

Let us also supplement Proposition 2.6 with the following:

Corollary 2.10: $\bar{\mathcal{Q}}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$ iff $\bar{\mathcal{Q}}_{n+1}(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

Proof: (\rightarrow) Since $\bar{\mathcal{Q}}_{n+1}(\epsilon) = [\bar{B}(\epsilon) \mid \bar{A}(\epsilon)\bar{\mathcal{Q}}_n(\epsilon)]$, and $\bar{\mathcal{Q}}_n(\epsilon)$ is full row rank at $\epsilon=0$, $\bar{A}(\epsilon)$ and $\bar{B}(\epsilon)$ are over $\mathbb{R}^n[[\epsilon]]$. Thus, $\bar{\mathcal{Q}}_\infty(\epsilon)$ is over $\mathbb{R}[[\epsilon]]$.

(\leftarrow) Trivial. ..

The standard form will prove to be very useful to us, especially for finding feedback to place eigenvalues (Section III). In the Appendix we develop an algorithm to get to a standard form without first constructing the reachability matrix and then explicitly determining its Smith decomposition in order to obtain the transformation matrix $P(\epsilon)$. The algorithm works directly on the pair $(A(\epsilon), B(\epsilon))$, and is a natural extension of the recommended procedure [3] for testing reachability of a constant pair (A, B) .

II.4 CONTINUOUS TIME

The natural counterpart to Definition 2.2 for continuous time is as follows:

Definition 2.11: $x \in \mathbb{R}^n[[\epsilon]]$ is ϵ^j -reachable if $\exists \tau \in \mathbb{R}^+$ and $u(t) \in 1/\epsilon^j \mathbb{R}^m[[\epsilon]] \forall t \in [0, \tau]$ such that $x(\tau) = x$, with $x(0) = 0$. Let \mathcal{X}^j be the set of all ϵ^j -reachable states, then $\mathcal{X}^0 \subset \mathcal{X}^1 \subset \mathcal{X}^2 \subset \dots$ and \mathcal{X}^j is an $\mathbb{R}[[\epsilon]]$ -submodule of $\mathbb{R}^n[[\epsilon]]$. We term \mathcal{X}^j the ϵ^j -reachable submodule. \dots

These submodules have properties analogous to those of discrete time for proper systems, as the following proposition and corollary show (the proofs are given in detail in [22]):

Proposition 2.12: Given a continuous time proper system described by the pair $(A(\epsilon), B(\epsilon))$, then $\mathcal{X}^0 = \langle A(\epsilon) | \mathcal{B}_\epsilon \rangle \cap \mathbb{R}^n[[\epsilon]]$ where $\langle A(\epsilon) | \mathcal{B}_\epsilon \rangle \equiv \sum_1^n A^{i-1}(\epsilon) \mathcal{B}_\epsilon$ and \mathcal{B}_ϵ is the image of $B(\epsilon)$ over $\mathbb{R}[[\epsilon]]$. \dots

Corollary 2.13: $\mathcal{X}^0 = P(\epsilon)D(\epsilon)\mathbb{R}^n[[\epsilon]]$ where $\mathcal{C}(\epsilon) = P(\epsilon)D(\epsilon)Q(\epsilon)$ is a Smith decomposition for the reachability matrix. \dots

Using the iterative relation $\mathcal{X}^{j+1} = \frac{1}{\epsilon} \{ \mathcal{X}^j \cap \epsilon \mathbb{R}^n[[\epsilon]] \}$, (Proposition 2.2), we can recover all the other reachability submodules from the Smith decomposition of the reachability matrix and Corollary 2.13. Therefore, all our results for discrete time also hold for continuous time.

III. SHIFTING EIGENVALUES BY $O(1)$ USING FULL STATE FEEDBACK

In this section, we restrict our attention to systems over $\mathbb{R}[[\epsilon]]$. These systems are proper and all eigenvalues of $A(\epsilon)$ are $O(1)$. We address the problem of arbitrarily shifting these eigenvalues by $O(1)$, using full state feedback. In other words, we wish to find $F(\epsilon)$ over $\mathbb{R}((\epsilon))$ such that $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$ has the desired eigenvalues at $\epsilon=0$.

Example 3.1: The eigenvalues of $A(\epsilon)$ in Example 2.1 are at $1+O(\epsilon)$ and $2+O(\epsilon)$. A state feedback of $[2 \ 4]$ shifts these eigenvalues to $3+O(\epsilon)$ and $2+O(\epsilon)$. It is not hard to see that there is no $O(1)$ state feedback that can move the eigenvalue at $2+O(\epsilon)$ by $O(1)$. However, a state feedback gain of $[5 \ -1/\epsilon]$ shifts the eigenvalues to $3+O(\epsilon)$ and $4+O(\epsilon)$. Here both eigenvalues are moved by $O(1)$, but an $O(1/\epsilon)$ feedback gain has to be used. Note that the closed loop system

$$A_F(\epsilon) = \begin{bmatrix} 6 & 1-1/\epsilon \\ 6\epsilon & 1 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

is not over $\mathbb{R}[[\epsilon]]$ but it is ϵ -reachable with the same indices, $n_0=1$ and $n_1=1$, as the original system, and is in proper standard form.

..

We shall now show that, for systems over $\mathbb{R}[[\epsilon]]$, the order of feedback gain necessary and sufficient to move all eigenvalues by $O(1)$ is directly given by the order of reachability of the system. Let us start by looking at ϵ^0 -reachable systems. In all that follows, Λ denotes a self-conjugate set of n eigenvalues, $\sigma(A)$

denotes the spectrum of A , and Z denotes the set of all integers.

Define

$$\alpha = \min_{r \in Z} \{r \mid \forall \Lambda, \exists F(\epsilon), O(1/\epsilon^r), \text{ s.t. } \sigma(A(\epsilon) + B(\epsilon)F(\epsilon))|_{\epsilon=0} = \Lambda\} \quad (3.1)$$

Hence α is the smallest order of feedback gain that will produce arbitrary $O(1)$ eigenvalue placement.

Proposition 3.2: The pair $(A(\epsilon), B(\epsilon))$, over $\mathbb{R}[[\epsilon]]$, is ϵ^0 -reachable iff $\alpha=0$.

Proof: (\rightarrow) If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0 -reachable, then, $\mathcal{C}(\epsilon)|_{\epsilon=0}$ has full row rank. Thus, the pair $(A(0), B(0))$ is reachable, and $\forall \Lambda, \exists F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t. $\sigma(A(\epsilon) + B(\epsilon)F)|_{\epsilon=0} = \sigma(A(0) + B(0)F) = \Lambda$. Hence $\alpha \leq 0$. Now assume $\alpha < 0$. Then, $\lim_{\epsilon \downarrow 0} F(\epsilon) = 0$ for those $F(\epsilon)$ of $O(1/\epsilon^\alpha)$ that produce arbitrary $O(1)$ eigenvalue placement according to (3.1). But then $\lim_{\epsilon \downarrow 0} (A(\epsilon) + B(\epsilon)F(\epsilon)) = A(0)$, so no $O(1)$ eigenvalues are moved, which is a contradiction. We conclude that $\alpha=0$.

(\leftarrow) Conversely, assume that $\alpha=0$, then $\forall \Lambda, \exists F=F(\epsilon)|_{\epsilon=0}$ s.t. $\sigma(A(0) + B(0)F) = \Lambda$. Thus, the pair $(A(0), B(0))$ is reachable, and $\mathcal{C}(\epsilon)|_{\epsilon=0}$ has full row rank, so the pair $(A(\epsilon), B(\epsilon))$ is ϵ^0 -reachable. \dots

Proposition 3.3: The pair $(A(\epsilon), B(\epsilon))$, over $\mathbb{R}[[\epsilon]]$, is ϵ^k -reachable iff $\alpha = k$.

Proof: (\rightarrow) If the pair $(A(\epsilon), B(\epsilon))$ is ϵ^k -reachable, then the pair $\bar{A}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)A(\epsilon)P(\epsilon)D(\epsilon)$, $\bar{B}(\epsilon) = D^{-1}(\epsilon)P^{-1}(\epsilon)$ is ϵ^0 -reachable and, by Lemma 2.8, is over $\mathbb{R}[[\epsilon]]$. Thus, by Proposition 3.2, $\forall \Lambda, \exists$ an $O(1)$ $\bar{F}(\epsilon)$ s.t. $\lambda(\bar{A}(\epsilon) + \bar{B}(\epsilon)\bar{F}(\epsilon))|_{\epsilon=0} = \Lambda$.

Let $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, then $F(\epsilon)$ is $O(1/\epsilon^k)$. By Lemma 2.4, the closed loop pair $(\bar{A}_{\bar{F}}(\epsilon), \bar{B}(\epsilon))$ is proper and so the coefficients of its characteristic equation are over $\mathbb{R}[[\epsilon]]$.

Thus,

$$\lim_{\epsilon \downarrow 0} \sigma(\bar{A}(\epsilon) + \bar{B}(\epsilon)\bar{F}(\epsilon)) = \lim_{\epsilon \downarrow 0} \sigma(A(\epsilon) + B(\epsilon)F(\epsilon)) \quad (3.2)$$

and $\alpha \leq k$. To see that the equality must hold, note first that

$$\bar{A}(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & \epsilon A_{0,1}(\epsilon) & \dots & \epsilon^k A_{0,k}(\epsilon) \\ A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & \epsilon^{k-1} A_{1,k}(\epsilon) \\ \vdots & \ddots & \ddots & \vdots \\ A_{k,0}(\epsilon) & A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \quad (3.3)$$

$n-n_{k-1}$ columns

Now, if $\alpha < k$, then the last $n-n_{k-1}$ columns of $\bar{F}(0) =$

$\lim_{\epsilon \downarrow 0} F(\epsilon)P(\epsilon)D(\epsilon) = 0$ for those $F(\epsilon)$ of $O(1/\epsilon^\alpha)$ that produce

arbitrary eigenvalue placement according to (3.1). But then

$$\lim_{\epsilon \downarrow 0} (\bar{A}(\epsilon) + \bar{B}(\epsilon)\bar{F}(\epsilon)) = \begin{bmatrix} * & 0 \\ * & A_{k,k}(0) \end{bmatrix} \quad (3.4)$$

where $*$ denotes some constant entries, and the eigenvalues corresponding to $A_{k,k}(\epsilon)$ are not moved by $O(1)$, which is a contradiction. We conclude that $\alpha = k$.

(\leftarrow) Clearly, the pair $(A(\epsilon), B(\epsilon))$ is ϵ^j -reachable for some j . By the first part of this proof, $\alpha = j$. Hence $j = k$ and the pair is ϵ^k -reachable. \dots

Note that if some pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$ is ϵ^0 -reachable then the closed loop pair $(A_{\bar{F}}(\epsilon), B(\epsilon))$, where $A_{\bar{F}}(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$, is ϵ^0 -reachable for all $F(\epsilon)$ of $O(1)$. Thus we have the following result:

Corollary 3.4: Given a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$, the ϵ^j -reachability indices n_i , as defined in Section II.3, are invariant under any feedback of the form $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$ where $\bar{F}(\epsilon)$ is $O(1)$. Also, the closed loop pair is proper. ..

The ϵ^j -reachable submodules of the standard form are uniquely determined by the indices, and the ϵ^j -reachable submodules of the original system are uniquely determined by the ϵ^j -reachable submodules of the standard form, via $P(\epsilon)$. Thus:

Corollary 3.5: Given a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$, the ϵ^j -reachability submodules are invariant under any feedback of the form $F(\epsilon) = \bar{F}(\epsilon)D^{-1}(\epsilon)P^{-1}(\epsilon)$, where $\bar{F}(\epsilon)$ is $O(1)$. ..

For the more general class of proper systems over $\mathbb{R}((\epsilon))$, the orders of feedback gains do not necessarily match the orders of reachability. Let us consider the following example:

Example 3.6: The pair

$$A(\epsilon) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\epsilon \\ 0 & 2\epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

corresponds to an ϵ -reachable system in proper standard form. Let

$$F(\epsilon) = \begin{bmatrix} f_1 & f_2 & 0 \\ f_3 & f_4 & 0 \end{bmatrix},$$

where the f_i are all scalar constants, then

$\det(\lambda I - A_F(\epsilon)) = \lambda^3 - (f_1 + f_4)\lambda^2 + (f_1 f_4 - f_2 f_3 - 2)\lambda + 2f_1$. Clearly, $f_i \in \mathbb{R}$ can be chosen appropriately to match any third degree polynomial with real coefficients. Therefore all eigenvalues of $A(\epsilon)$ can be arbitrarily moved by $O(1)$ using only $O(1)$ feedback gains. What

happens in this example is that an $O(1)$ gain for the third state component produces an $O(1/\epsilon)$ input for the second component. Therefore, even with $O(1)$ gains, the input values themselves will be $O(1/\epsilon)$, as would be expected when producing $O(1)$ shifts in eigenvalues for this ϵ -reachable system. ..

The overall effect of $O(1)$ feedback on the eigenvalues, even for systems over $\mathbb{R}[[\epsilon]]$, is a more subtle issue than the order of feedback necessary to shift the eigenvalues by $O(1)$. Consider the following example:

Example 3.7: Let

$$A(\epsilon) = \begin{bmatrix} 0 & 1 \\ \epsilon & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

The reachability indices are $n_0=1$ and $n_1=2$. The eigenvalues of $A(\epsilon)$ are at $\pm\sqrt{\epsilon}$. Feedback of $[-1 -1]$ moves the eigenvalues to -1 and $-\epsilon$. Thus, the effect of feedback is larger than $O(\epsilon)$, namely $O(\sqrt{\epsilon})$. (It is worth noting that the original system did not have well-behaved time scale structure in the sense of [1,2], and that the feedback produces well-behaved time scale structure.) ..

We leave these problems for further research. Chapter V suggests some potential extensions.

An extension of Algorithm A.3 can be used to compute the feedback matrix necessary to shift eigenvalues by some desired amount. Application of Algorithm A.3 produces a pair $(A_k(\epsilon), B_k(\epsilon))$, where $A_k(\epsilon) = S^{-1}(\epsilon)A(\epsilon)R(\epsilon)$, $B_k(\epsilon) = S^{-1}(\epsilon)B(\epsilon)$, where $(A_k(0), B_k(0))$ is reachable and $S(\epsilon)$ is the product of all

the similarity transformations used to achieve the final pair. From the pair $(A_k(0), B_k(0))$, we can compute a feedback matrix F such that the eigenvalues of $A_{kF}(0) = A_k(0) + B_k(0)F$ are as desired. We have that $\sigma(A_{kF}(\epsilon))|_{\epsilon=0} = \sigma(A_{kF}(0))$ and that $(A_{kF}(\epsilon), B(\epsilon))$ is proper. Let $F(\epsilon) = FS^{-1}(\epsilon)$ and $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$. Since $S(\epsilon)$ is invertible for $\epsilon \in (0, a)$ for some $a \in \mathbb{R}$, $(A_F(\epsilon), B(\epsilon))$ is also proper. Therefore, as in the proof of Proposition 3.3, the eigenvalues of $A_F(\epsilon)$ are as desired.

This algorithm was applied in [22] to a fifth order, weakly reachable system over \mathbb{R} with one input. The system was first parametrized by replacing certain small entries by $O(1)$ multiples of powers of ϵ . The feedback gain to place $O(1)$ eigenvalues calculated for the parametrized system by the above approach was evaluated at the specific value of ϵ corresponding to the original system. This approach produced far better numerical results than calculating the feedback directly for the given system. Similar concerns have been expressed by authors interested in numerical issues of multivariable pole placement for linear time invariant systems (as explained in I.1). Our approach would attempt to address those issues by scaling the pair (A, B) appropriately. Unfortunately, (A, B) has to be parametrized by ϵ first. Further study of this problem has been left for future research, though some heuristic suggestions for parametrizations are made in Section V.

IV. ALMOST INVARIANT SUBSPACES

IV.1 $(A(\epsilon), B(\epsilon))$ -INVARIANCE AND ALMOST (A, B) -INVARIANCE

In this section, we use our framework to provide some new insights on the notions of almost (A, B) -invariance and almost (A, B) -controllability, introduced by J. C. Willems [17] into the geometric approach to linear systems, [10].

To give a flavor for our approach, let us consider the following example:

Example 4.1: Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is easy to see from the results in [17] that $\mathcal{V}_a = \text{Im}[1 \ 0]'$ is an almost (A, B) -invariant subspace. Consider the family of subspaces, $\{\mathcal{V}_\epsilon\}$, generated by $[1 \ \epsilon]'$ for each fixed $\epsilon \in (0, \infty)$.

Since

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} (1/\epsilon) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-1/\epsilon),$$

these subspaces are (A, B) -invariant, [10]. As we let $\epsilon \rightarrow 0$, $\{\mathcal{V}_\epsilon\} \rightarrow \mathcal{V}_a = \text{Im}[1 \ 0]'$, which is an almost (A, B) -invariant subspace. So we have found a sequence of (A, B) -invariant subspaces (\mathcal{V}_ϵ) evaluated at different values of ϵ) that converge to an almost (A, B) -invariant subspace. Using the relation $(-1/\epsilon) = F(\epsilon)[1 \ \epsilon]'$ with $F(\epsilon) = [-1/\epsilon \ 0]$, these subspaces are $A_F(\epsilon)$ invariant where

$$A_F(\epsilon) = A - BF(\epsilon) = \begin{bmatrix} 1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, the $\{\mathcal{V}_\epsilon\}$ are coasting subspaces, [17], i.e. they are (A, B) -invariant but they have no (A, B) -controllable part, whereas

\mathcal{V}_a is a sliding subspace, [17], i.e. it is almost (A, B) -invariant but it has no (A, B) -invariant part.

Note that an eigenvalue of $A_F(\epsilon) \rightarrow +\infty$ as $\epsilon \rightarrow 0$. On the other hand, consider the family of (A, B) -invariant subspaces, $\{\mathcal{V}_\epsilon\}$, generated by $[1 \ -\epsilon]$. As $\epsilon \rightarrow 0$, $\{\mathcal{V}_\epsilon\} \rightarrow \mathcal{V}_a$ also. By going through the above procedure, we get $F'(\epsilon) = [1/\epsilon \ 0]$ and

$$A_F(\epsilon) = \begin{bmatrix} -1/\epsilon & 0 \\ 1 & 0 \end{bmatrix}.$$

Now the eigenvalue of $A_F(\epsilon)$ that blows up approaches $-\infty$ as $\epsilon \rightarrow 0$.

..

We proceed with proving some results related to the above observations, but we first state some algebraic properties that we use extensively.

$\mathbb{R}^n((\epsilon))$ is a vector space over the field $\mathbb{R}((\epsilon))$. Let \mathcal{V}_ϵ be a subspace of $\mathbb{R}^n((\epsilon))$ and let the columns of $V(\epsilon) = [v_1(\epsilon) | \dots | v_\mu(\epsilon)]$ be a linearly independent set that spans \mathcal{V}_ϵ . Since \mathcal{V}_ϵ is closed under multiplication by elements in $\mathbb{R}((\epsilon))$, it is possible to pick $v_i(\epsilon)$ such that $v_i(\epsilon) \in \mathbb{R}[[\epsilon]]$ and $V(0)$ has full column rank. Hence $V(\epsilon)$ has full column rank for small enough ϵ . Note that the span of the columns of $V(\epsilon)$, for any fixed ϵ , is a subspace of \mathbb{R}^n . Thus it is also possible to think of \mathcal{V}_ϵ as a sequence of subspaces of \mathbb{R}^n defined by $V(\epsilon)$ for different values of ϵ . We use this to connect our results to their counterparts in [17] and [10].

Definition 4.2: $\mathcal{V}_\epsilon \subset \mathbb{R}^n((\epsilon))$ is $(A(\epsilon), B(\epsilon))$ -invariant if $\exists F(\epsilon) : \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ s.t. $A_F(\epsilon)\mathcal{V}_\epsilon \subset \mathcal{V}_\epsilon$, where $A_F(\epsilon) = A(\epsilon) + B(\epsilon)F(\epsilon)$.

..

We denote the family of $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspaces by \underline{V}_ϵ . In some cases, we shall consider $(A(\epsilon), B(\epsilon))$ -invariant $\mathbb{R}((\epsilon))$ -subspaces for $A(\epsilon)=A$ and $B(\epsilon)=B$. We use the same notation, and assume that the reader will infer the relevant underlying system from the context. Following Willems [17], we denote the family of (A, B) -invariant subspaces by \underline{V} and the family of almost (A, B) -invariant subspaces by \underline{V}_a .

A straightforward extension of this definition is the following well known result [10]:

Proposition 4.3: $\underline{V}_\epsilon \in \underline{V}_\epsilon$ iff $A(\epsilon)\underline{V}_\epsilon \subset \underline{V}_\epsilon + \mathcal{B}$, where $\mathcal{B} = B(\epsilon)\mathbb{R}^m((\epsilon))$. ..

Definition 4.4: Given $\underline{V}_a \subset \mathbb{R}^n$ and $\underline{V}_\epsilon \subset \mathbb{R}^n((\epsilon))$, $\underline{V}_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \underline{V}_a$ if whenever $\{v_1(\epsilon), \dots, v_\mu(\epsilon)\}$, $v_i(\epsilon) \in \mathbb{R}^n[[\epsilon]]$, is a basis for \underline{V}_ϵ , the set of vectors $\{v_1(0), \dots, v_\mu(0)\}$ forms a basis for \underline{V}_a (this is convergence in the Grassmannian sense). ..

One can always construct a matrix $W(\epsilon)$ over $\mathbb{R}[[\epsilon]]$ such that $W(0) = I$ and $v_i(\epsilon) = W(\epsilon)v_i(0)$. Thus an alternate representation of \underline{V}_ϵ would be $W(\epsilon)\underline{V}_a$.

The following result enables us to establish a connection between our framework and the notion of almost (A, B) -invariance. It provides a method to compute approximations for the distributional inputs required to steer the trajectories of an almost (A, B) -invariant subspace exactly through that subspace. Using these high gain feedback approximations one can steer

trajectories arbitrarily close to an almost (A, B) -invariant subspace.

Proposition 4.5: For a given pair (A, B) , if $\gamma_a \in \underline{V}_a$ then $\exists \gamma_\epsilon \in \underline{V}_\epsilon$ such that $\gamma_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \gamma_a$. ..

The proof is very similar in principle to that of Willems [17] and it is given in detail in [22]. However, note that the converse of the above proposition does not hold, though [17] claims that it does. To illustrate this, consider the following example:

Example 4.6: Let

$$A = \begin{bmatrix} 0_3 & 0_3 \\ I_3 & 0_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix}.$$

Consider $\gamma = \{v_1(\epsilon), v_2(\epsilon), v_3(\epsilon)\}$ where $v_1(\epsilon) = [1 \ 0 \ 0 \ 0 \ \epsilon \ 0]', v_2(\epsilon) = [0 \ 0 \ 0 \ 1 \ 0 \ 0]', v_3(\epsilon) = [0 \ 1 \ 0 \ 0 \ 0 \ 1]'$ and $\{\cdot\}$ denotes span over $\mathbb{R}((\epsilon))$. $\gamma \in \underline{V}_\epsilon$ and $\gamma \xrightarrow[\epsilon \rightarrow 0]{} \mathcal{L}$ where $\mathcal{L} = \{v_1(0), v_2(0), v_3(0)\}$ and $\{\cdot\}$ denotes span over \mathbb{R} . But \mathcal{L} is not an almost (A, B) -invariant subspace. ..

Willems [17] poses the problem of finding an input that steers the trajectories of a system arbitrarily close to an almost (A, B) -invariant subspace. Our approach shows how this can be done. We show below how to construct an (A, B) -invariant $\mathbb{R}((\epsilon))$ -subspace that approaches the almost (A, B) -invariant subspace in the Grassmannian sense. The desired input then follows on calculating the feedback that makes the (A, B) -invariant $\mathbb{R}((\epsilon))$ -subspace $A_F(\epsilon)$ -invariant.

Recall from [17] that any almost (A, B) -invariant subspace \mathcal{V}_a can be represented as $\mathcal{V}_a = \mathcal{V} + \mathcal{R}_a$ where \mathcal{V} is (A, B) -invariant and \mathcal{R}_a is almost (A, B) -controllable and any almost (A, B) -controllability subspace \mathcal{R}_a can be represented as $\mathcal{R}_a = \mathcal{R}_o \oplus \mathcal{R}_s$ where \mathcal{R}_o is the supremal (A, B) -controllability subspace in \mathcal{R}_a and \mathcal{R}_s is a sliding subspace. By a construction in the proof of Proposition 4.5 in [22], we can find $\mathcal{V}_c \in \mathcal{V}_\epsilon$ where $\mathcal{V}_c = Q(\epsilon)\mathcal{R}_s$, $Q(\epsilon)$ over $\mathbb{R}[[\epsilon]]$ and $Q(0) = I$, where \mathcal{V}_c is a coasting $\mathbb{R}((\epsilon))$ -subspace whose associated eigenvalues approach $-\infty$ as $\epsilon \rightarrow 0$. The feedback $F(\epsilon)$ that makes \mathcal{V}_c an $A_F(\epsilon)$ -invariant $\mathbb{R}((\epsilon))$ -subspace can be calculated and provides the desired input. Those eigenvalues of $A_F(\epsilon)$, for fixed ϵ , that correspond to \mathcal{R}_s approach $-\infty$ as $\epsilon \rightarrow 0$. This increases the magnitude of the feedback gains, and the generated inputs and their derivatives approach impulses in the limit. The eigenvalues corresponding to \mathcal{R}_o can be assigned by the usual pole placement methods.

As an illustration of the procedure, consider the following example, which contains the essential features of the general case:

Example 4.7: Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{V}_a = \{\bar{v}_1, \bar{v}_2\}, \quad \text{where } \bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

\mathcal{V}_a is an almost (A, B) -invariant subspace, and in fact it is a sliding subspace. Consider $\mathcal{V}_\epsilon = \{v_1(\epsilon), v_2(\epsilon)\}$, where $v_1(\epsilon) = [1 \ -\epsilon \ \epsilon^2]'$ and $v_2(\epsilon) = [0 \ 1 \ -2\epsilon]'$. Note that \mathcal{V}_ϵ is a coasting $\mathbb{R}((\epsilon))$ -subspace, i.e. it is (A, B) -invariant but not (A, B) -controllable. Furthermore, $v_1(0) = \bar{v}_1$, $v_2(0) = \bar{v}_2$ and $\mathcal{V}_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \mathcal{V}_a$. Also, $v_i(\epsilon) = P(\epsilon)\bar{v}_i$, for $i=1, 2$, where

$$P(\epsilon) = \begin{bmatrix} 1 & 0 & 0 \\ -\epsilon & 1 & 0 \\ 2\epsilon & -\epsilon^2 & 1 \end{bmatrix}$$

gets its lower triangular entries from a Pascal triangle

construction with alternating signs (see [22]). Solving the

equations $A(\epsilon)[v_1(\epsilon) | v_2(\epsilon)] = [v_1(\epsilon) | v_2(\epsilon)]g_v(\epsilon) + Bg_u(\epsilon)$ and

$g_u(\epsilon) = F(\epsilon)[v_1(\epsilon) | v_2(\epsilon)]$ yields $F(\epsilon) = [2/\epsilon \ 1/\epsilon^2 \ 0]$, and

$$A_F(\epsilon) = A - BF(\epsilon) = \begin{bmatrix} -2/\epsilon & -1/\epsilon^2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with γ_ϵ being $A_F(\epsilon)$ -invariant. Note that the desired input $u(t) = -F(\epsilon)x(t)$. On the other hand, the eigenvalues of $A_F(\epsilon)$ that correspond to γ_a , for fixed ϵ , are both at $-1/\epsilon$. They are stable and approach $-\infty$ as $\epsilon \rightarrow 0$. \dots

IV.2 $(A(\epsilon), B(\epsilon))$ -CONTROLLABILITY AND ALMOST (A, B) -INVARIANCE

We now proceed with the notion of $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces, adopting Wonham's definition [10] of (A, B) -controllability subspaces:

Definition 4.8: $\mathcal{R} \in \mathbb{R}^n((\epsilon))$ is an $(A(\epsilon), B(\epsilon))$ -controllability subspace if there exist maps $F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ and $G(\epsilon): \mathbb{R}^m((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ such that $\mathcal{R} = \langle A(\epsilon) + B(\epsilon)F(\epsilon) | \text{Im}(B(\epsilon)G(\epsilon)) \rangle$. \dots

We denote the family of $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces by \underline{R}_ϵ . For some cases, we consider $(A(\epsilon), B(\epsilon))$ -controllability $\mathbb{R}((\epsilon))$ -subspaces for $A(\epsilon)=A$ and $B(\epsilon)=B$. The same notation is also used for these cases. Following [17] we denote the family of (A, B) -controllability subspaces by \underline{R} and the family of almost (A, B) -controllability

subspaces by \underline{R}_a .

To put the above definition into a more usable form, consider the following proposition, which simply restates results of Wonham [10] in the present framework:

Proposition 4.9: (a). $\mathcal{R} \in \underline{R}_\epsilon$ iff there exists a map $F(\epsilon): \mathbb{R}^n((\epsilon)) \rightarrow \mathbb{R}^m((\epsilon))$ such that $\mathcal{R} = \langle A(\epsilon) + B(\epsilon)F(\epsilon) | \mathcal{B} \cap \mathcal{R} \rangle$ where \mathcal{B} represents the range of $B(\epsilon)$ over $\mathbb{R}((\epsilon))$.

(b). $\mathcal{R} = \langle A_F(\epsilon) | \mathcal{B} \cap \mathcal{R} \rangle$ for every map $F(\epsilon) \in F(\mathcal{R})$. where $F(\mathcal{R})$ represents the family of feedback matrices $F(\epsilon)$ such that \mathcal{R} is $A_F(\epsilon)$ -invariant.

Proof: The proofs are very similar to that of Wonham [10] and they are given in [22].

..

Let $\mathcal{R}_\epsilon \in \underline{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \mathcal{R}_n$. Then, it turns out that \mathcal{R}_n is almost (A,B)-invariant. Finding inputs for steering trajectories arbitrarily close to \mathcal{R}_n is done by calculating an $F(\epsilon)$ such that \mathcal{R}_ϵ is $A_F(\epsilon)$ -invariant and the eigenvalues corresponding to \mathcal{R}_ϵ are $O(1)$ and asymptotically stable. The following lemma and proposition show this:

Lemma 4.10: Given a pair (A,B), let $\mathcal{R}_\epsilon \in \underline{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \mathcal{R}_n$, then \forall $O(1)$ x_0 s.t. $d(x_0, \mathcal{R}_n)$ is $O(\epsilon)$ and $\forall \tau > 0$, \exists an input function $u(t)$ s.t. $d(x_0(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$, where $x_0(t, \epsilon)$ is the trajectory defined by $u(t)$ and the initial condition x_0 .

Proof: Here we first need to find a trajectory in \mathcal{R}_ϵ which is $O(1)$

for $0 < t < \tau$. Find $F(\epsilon)$ s.t. \mathcal{R}_ϵ is $A_F(\epsilon)$ -invariant and the eigenvalues of $A_F(\epsilon)$ corresponding to \mathcal{R}_ϵ are all $O(1)$ and asymptotically stable. Then $\forall O(1) x_1 \in \mathcal{R}_\epsilon$, $x_1(t, \epsilon) \in \mathcal{R}_\epsilon \forall t > 0$ where $x_1(t, \epsilon)$ is the trajectory defined by the initial condition x_1 and the input specified by $F(\epsilon)x(t)$. Since the eigenvalues of $A_F(\epsilon)$ corresponding to \mathcal{R}_ϵ are all $O(1)$ and stable, $x_1(t, \epsilon)$ is also $O(1)$. Therefore, $d(x_1(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$, since the 'angle' between \mathcal{R}_ϵ and \mathcal{R}_n is $O(\epsilon)$. Consider $x_2(t, \epsilon)$, the trajectory defined by the initial condition $x_2 = x_0 - x_1$, with $x_1 \in \mathcal{R}_\epsilon$ chosen such that x_2 is $O(\epsilon)$. Since the eigenvalues of $A_F(\epsilon)$ are $O(1)$, $\forall \tau > 0$ $x_2(t, \epsilon)$ is $O(\epsilon)$ for $0 < t \leq \tau$. Thus, $d(x_0(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$. \dots

Proposition 4.11: Given a pair (A, B) , let $\mathcal{R}_\epsilon \in \mathcal{R}_\epsilon$ and $\mathcal{R}_\epsilon \xrightarrow[\epsilon \rightarrow 0]{} \mathcal{R}_n$, then $\mathcal{R}_n \in \mathcal{V}_a$.

Proof: Pick some $\tau > 0$ and apply Lemma 4.10. Thus, $\exists u(t)$ s.t. $d(x(t, \epsilon), \mathcal{R}_n)$ is $O(\epsilon)$ for $0 < t \leq \tau$. Then $\exists \epsilon_0 > 0$ s.t. $d(x(t, \epsilon), \mathcal{R}_n) < \delta$ for $0 < t \leq \tau$ and $\forall \epsilon \leq \epsilon_0$. Use $x(\tau, \epsilon)$ as the initial condition to reapply Lemma 4.10 for the interval $\tau < t \leq 2\tau$. Find $\epsilon_1 > 0$ s.t. $\epsilon_1 \leq \epsilon_0$ and $d(x(t, \epsilon_1), \mathcal{R}_n) < \delta$ for $\tau < t \leq 2\tau$. Repeated use of Lemma 4.10 achieves the desired result. \dots

To illustrate these, consider the following example:

Example 4.12: Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{R}_\epsilon = \text{Im} \begin{bmatrix} 1 \\ \epsilon \\ 0 \end{bmatrix} + \text{Im} \begin{bmatrix} 0 \\ 0 \\ \epsilon \end{bmatrix}$$

Note that $\mathcal{R}_n = \text{Im}[1 \ 0 \ 0]' + \text{Im}[0 \ 0 \ 1]'$ and it is an almost (A, B) -invariant subspace. Let $F(\epsilon) = [-3 \ 0 \ -2/\epsilon]$, then \mathcal{R}_ϵ is $A_F(\epsilon)$ -invariant and the eigenvalues corresponding to \mathcal{R}_ϵ are at -2 ,

-4, asymptotically stable and $O(1)$. Pick the initial state x_0 of Lemma 4.10 as $x_0 = [1 \ 0 \ 0]'$. Let $x_1 = [1 \ \epsilon \ 0]'$ $\in \mathcal{R}_\epsilon$. Then, $x_1(t, \epsilon) = [-e^{-t} + 2e^{-2t} \ -\epsilon e^{-t} + 2\epsilon e^{-2t} \ -\epsilon e^{-t} - \epsilon e^{-2t}]'$ $\in \mathcal{R}_\epsilon$, and $d(x_1(t, \epsilon), \mathcal{R}_n)$ is clearly $O(\epsilon)$ for any finite τ . On the other hand, $x_2 = [0 \ -\epsilon \ 0]'$ and $x_2(t, \epsilon) = [2\epsilon e^{-2t} \ 2\epsilon^2 e^{-2t} \ -\epsilon^2 e^{-2t}]'$. Thus, $d(x_0(t), \mathcal{R}_n)$ is $O(\epsilon)$. So, in the spirit of Proposition 4.11, this may be bounded by any δ for any given τ by picking an appropriate $\epsilon = \epsilon_0$. Then, using $x(\tau, \epsilon)$ as the new initial state and repeated use of this procedure achieves the desired result. ..

In this section, we examined the notions of almost (A, B) -invariant and almost (A, B) -controllability subspaces in the framework that we have developed in this paper and [22]. We outlined a method for calculating inputs that steer trajectories arbitrarily close to almost (A, B) -invariant subspaces or equivalently force the eigenvalues corresponding to sliding parts of almost (A, B) -controllability subspaces to approach $-\infty$. We also analyzed the properties of limits of elements in \mathcal{V}_ϵ and \mathcal{R}_ϵ as $\epsilon \downarrow 0$ from a trajectory point of view.

V. CONCLUSIONS

In this paper, we have developed an algebraic approach to high gain controls for linear dynamic systems with varying orders of reachability. Based on this approach, we addressed the issues of high gain inputs for reaching target states, high gain feedback for pole placement and high gain inputs for steering trajectories arbitrarily close to almost (A, B) -invariant subspaces and almost (A, B) -controllability subspaces.

The results presented here suggest several direction for further research. It is of interest to analyze the orders of feedback gains for shifting eigenvalues by $O(1)$ in the more general case of proper systems, rather than just systems over $\mathbb{R}[[\epsilon]]$. Intuitively, if a mode is ϵ -reachable but " $1/\epsilon$ -observable", in that it has a $1/\epsilon$ coupling to other states, then it should be possible to shift its eigenvalue by $O(1)$ using $O(1)$ feedback gain. A related problem is that of changing the dynamics of a given continuous time system that has multiple time scales [1],[2] without changing its time scale structure. This would involve shifting $O(\epsilon^j)$ eigenvalues only by $O(\epsilon^j)$ rather than $O(1)$.

A key problem that bears attention is that of parametrizing systems over \mathbb{R} . Two heuristic methods could be suggested for this. One is to recognize small entries in the matrix, either isolated or added to another entry, and replace these with powers of ϵ . Another method for parametrization could come from

numerical reachability tests [3], where for example small singular values at different stages of a test may be replaced by (appropriate powers of) ϵ .

It will be important to develop dual results for systems with observations $y[k] = C(\epsilon)x[k]$ or $y(t) = C(\epsilon)x(t)$. This could then lead to research on connections to optimal control, realization theory and especially to the work on balanced realizations, [8].

APPENDIX

Here we develop an algorithm to recover a standard form without forming the reachability matrix and computing its Smith decomposition. The proofs and details on the algorithm are presented in [22]. Our algorithm can only deal with a pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$, so this restriction is assumed here. Then, the structure of a pair $(A(\epsilon), B(\epsilon))$ in standard form is as follows:

$$A(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \dots & A_{0,k}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & A_{1,k}(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^k A_{k,0}(\epsilon) & \epsilon^{k-1} A_{k,1}(\epsilon) & \dots & A_{k,k}(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_k \} \quad (A.1a)$$

$$B(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^k B_k(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_k \} \quad (A.1b)$$

Proposition A.1 : An ϵ^k -reachable pair $(A(\epsilon), B(\epsilon))$ over $\mathbb{R}[[\epsilon]]$ is in proper standard form with indices p_0, \dots, p_k iff $A(\epsilon)$ and $B(\epsilon)$ satisfy the following condition: Let $F_i(\epsilon) = D_i^{-1}(\epsilon)A(\epsilon)D_i(\epsilon)$, $G_i(\epsilon) = D_i^{-1}(\epsilon)B(\epsilon)$ where $D_i(\epsilon) = \text{diag}\{I_{p_0}, \dots, \epsilon^{i-1}I_{p_i}, \dots, \epsilon^{k-1}I_{p_k}\}$ then the reachable subspace of $(F_i(0), G_i(0))$ is $\mathcal{E}_i = \text{Im} \begin{bmatrix} I_{n_i} \\ 0 \end{bmatrix}$, for $\forall i \in [0 \dots k]$.

Definition A.2: Let

$$\bar{A}_i(\epsilon) = \begin{bmatrix} A_{0,0}(\epsilon) & A_{0,1}(\epsilon) & \dots & A_{0,i}(\epsilon) \\ \epsilon A_{1,0}(\epsilon) & A_{1,1}(\epsilon) & \dots & A_{1,i}(\epsilon) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^i A_{i,0}(\epsilon) & \epsilon^{i-1} A_{i,1}(\epsilon) & \dots & A_{i,i}(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_i\} \quad (A.2a)$$

$$\bar{B}_i(\epsilon) = \begin{bmatrix} B_0(\epsilon) \\ \epsilon B_1(\epsilon) \\ \vdots \\ \epsilon^i B_i(\epsilon) \end{bmatrix} \{p_0, p_1, \dots, p_i\} \quad (A.2b)$$

then $(\bar{A}_i(\epsilon), \bar{B}_i(\epsilon))$ is the ϵ^i -reachable subsystem of $(A(\epsilon), B(\epsilon))$ with indices n_0, \dots, n_i . \dots

Similar to the submodule structure, the ϵ^i -reachable subsystem contains all ϵ^j -reachable subsystems for $j = 0, \dots, i-1$. The subsystems are layered with weak couplings of different orders of ϵ between each component, as shown in Figure A. Also,

$$\mathcal{C}_i(\epsilon) \mathbb{R}^{mn_i}[[\epsilon]] \oplus \epsilon^{i+1} \mathbb{R}^{n-n_i}[[\epsilon]] \supset \mathcal{Y}^0 \quad (A.3)$$

and the sequence $\{\mathcal{C}_i(\epsilon) \mathbb{R}^{mn_i}[[\epsilon]] \oplus \epsilon^{i+1} \mathbb{R}^{n-n_i}[[\epsilon]]\}$ converges to \mathcal{Y}^0 in k steps. In other words, ϵ^0 -reachable submodules of the ϵ^i -reachable subsystems approximate the ϵ^0 -reachable submodule of the system in standard form upto ϵ^{i+1} accuracy. We use this in Algorithm A.3 below.

Computation of the reachability matrix is very expensive. One has to calculate $A^i(\epsilon)B(\epsilon)$ for all the terms in the expansions of $A(\epsilon)$ and $B(\epsilon)$. Thus, it is desirable to work directly with the pair $(A(\epsilon), B(\epsilon))$. The following algorithm takes advantage of

Proposition A.1 to recover the ϵ^j -reachability indices. At every step, the reachable subspace of a pair, evaluated at $\epsilon=0$, is computed. Then the pair is updated by an appropriate scaling of the unreachable part by $1/\epsilon$. The algorithm uses the coefficients of the Taylor expansions of the higher order terms only when necessary. Also, it is possible to recover the actual Smith decomposition of the reachability matrix from the algorithm, if the transformations used in the algorithm are restricted to be permutation matrices and lower triangular matrices, though this restriction compromises numerical stability (see [22]).

Algorithm A.3:

Initialize: $A_0(\epsilon) = A(\epsilon)$, $B_0(\epsilon) = B(\epsilon)$, $i = 0$

Step i:

1. Find T_i such that

$$T_i^{-1}A_i(0)T_i = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}^{n_i}, \quad T_i^{-1}B_i(0) = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}^{n_i}$$

with (A_1, B_1) reachable. This determines n_i .

2. If $n_i = n$ then go to End, else continue.

3. Let $A_{i+1}(\epsilon) = D_i^{-1}(\epsilon)T_i^{-1}A_i(\epsilon)T_iD_i(\epsilon)$, $B_{i+1} = D_i^{-1}(\epsilon)T_i^{-1}B_i(\epsilon)$ where $D_i(\epsilon) = \text{diag}\{I_{n_i}, \epsilon I_{n-n_i}\}$.

(It is not necessary to carry out the computation for all the coefficients of $A_i(\epsilon)$ and $B_i(\epsilon)$; see Note 1 in [22].)

4. Increment i , go to Step i.

End: $k = i$, the system is ϵ^k -reachable with indices n_0, \dots, n_k .

..

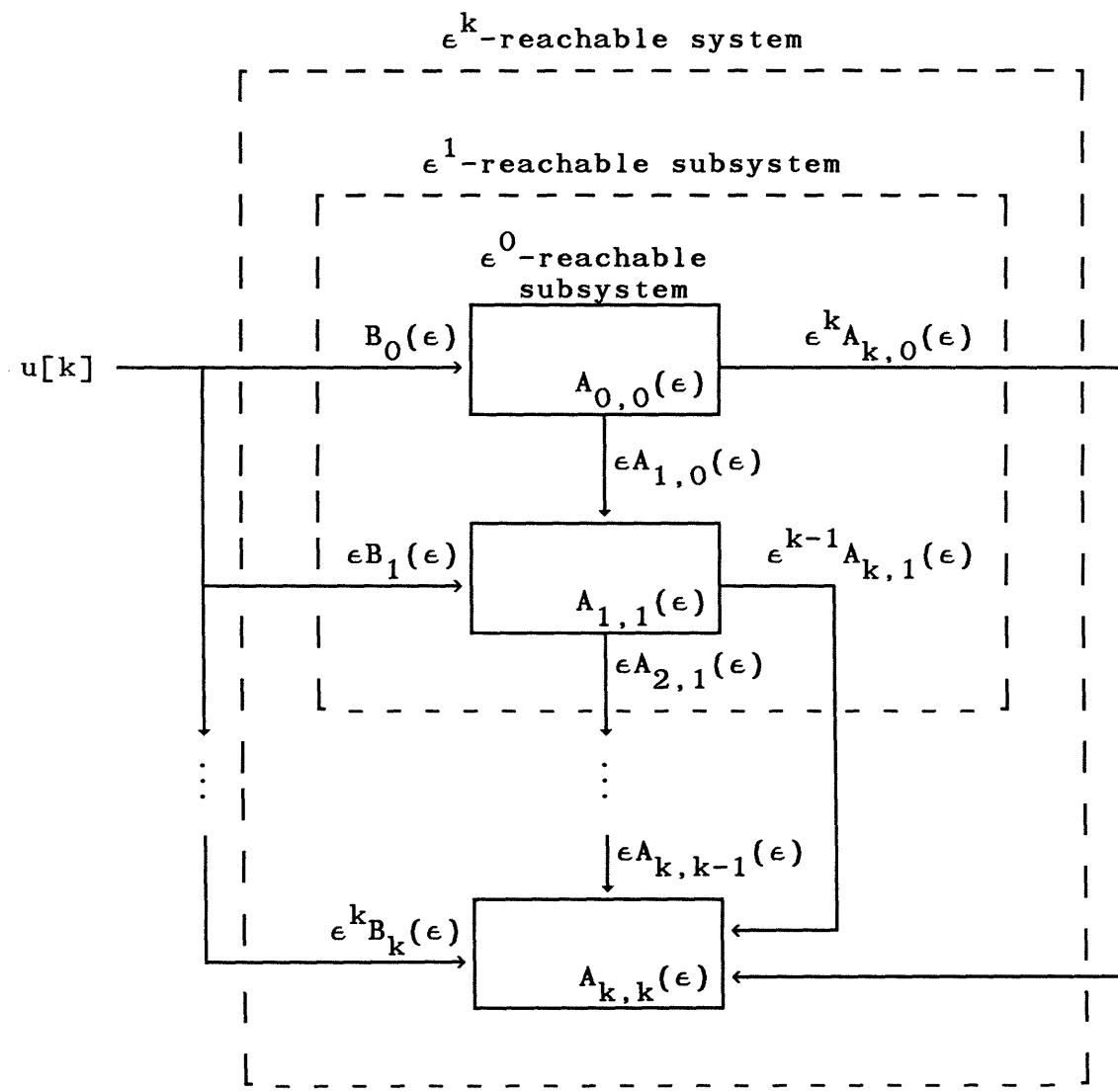


FIGURE A.1 Block diagram showing the structure of an ϵ^k -reachable system in standard form (upper off-diagonal blocks of $A_s(\epsilon)$ have been omitted for clarity).

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